Vacuum Solutions of Classical Gravity on Cyclic Groups from Noncommutative Geometry

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Abstract

Based on the observation that the moduli of parallel transports(link variable) on a cyclic group modify the metric on this group, we construct several action functionals for these parallel transports within the framework of noncommutative geometry. After solving the equations of motion, we find that one type of action can give nontrivial vacuum solutions for gravity on this cyclic group in a broad range of coupling constants and that such solutions can be expressed with Chebyshev's polynomials.

Keywords: parallel transport, Connes' distance formula, gravity, vacuum solution, noncommutative geometry, cyclic group

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I. INTRODUCTION

A striking feature in noncommutative-geometry(NCG)-based field-theoretic model-buildings is that the bosonic sector of a physical system can be determined by a generalized Dirac operator in fermionic sector through a geometric construction. As for Yang-Mills fields, the most successful example is the reconstruction of standard model of particle physics from Connes' NCG [1]; in addition, Wilson action of lattice gauge theory has been deduced in our recent work under the same NCG formalism [2]. As to gravity, gravitational fields are coupled naturally to gauge fields by the *spectral action principle* [3] [4]. However, compared with the simplicity and tidiness of NCG manipulation of gauge theory, gravity is far from being manifest and easy to handle. In our understanding, more exercises on the latter are needed.

At the end of [2], we discussed that the effect of non-unitary 1D parallel transports on Connes' distance of a one-dimensional lattice is to flex the linear(Euclidean)-distance with the moduli of these parallel transports. This feature has already indicated the general feature of NCG that gauge-connection and metric are easily interwound on a noncommutative space. In fact, we can interpret that the arguments of these parallel transports play the role of a $U(1)$ gauge connection while that the moduli of these parallel transports play the role of a linear connection. Basing on this observation, in this contribution, we will explore the (classical) gravitational physics of this 1D lattice in depth. We try to construct action functionals for gravitational fields(bosonic) also from Dirac operator, deduce the equations of motion, then solve them and give a series of vacuum solutions. It will be showed that in a broad range of coupling constants, there will be non-trivial vacuum solutions that can be expressed with Chebyshev's polynomials and break translation invariance. Of course, this model is far from being a realistic description of gravitational system in physical world. The significance to treat it in such a scrutiny is that it provides an easily-handling example to formulate gravity theory on discrete sets. Though being simple, it has shown a lot of general features of NCG approach to gravity problems.

This paper is organized as the following way. In section II, NCG on a cyclic group is formulated, especially Dirac operator twisted with parallel transports is defined. Several action functionals are established in section III, in which two will be discussed in length. Three types of vacua will be found out, in which only the last one is nontrivial. Some open discussions are put into section IV.

II. KINEMATICS OF DIRAC OPERATOR ON CYCLIC GROUPS

Let $\mathcal{Z}_N$ be a N-order cyclic group $\mathcal{Z}_N = \{0, 1, 2, \ldots, N-1\}$ whose multiplication rule is just integer addition modulo N. Most addition operations appearing below are understood as additions in $\mathcal{Z}_N$. $\mathcal{A}(\mathcal{Z}_N)$ is the algebra of complex functions on $\mathcal{Z}_N$ and there is a *regular representation* of $\mathcal{Z}_N$ on $\mathcal{A}(\mathcal{Z}_N)$ generated by $(T^+f)(x) = f(x+1), \forall f \in \mathcal{A}(\mathcal{Z}_N), x \in \mathcal{Z}_N$. Introduce $End_{\mathcal{C}}(L))$ to be the algebra of complex-linear transformations on a complex linear space L; then $T^+ \in End_{\mathcal{C}}(\mathcal{A}(\mathcal{Z}_N))$. We will use Sp to denote the trace on $End_{\mathcal{C}}(\mathcal{A}(\mathcal{Z}_N))$ and $\mathbf{1}$ for the identity transformation below. For any finite N, $(T^+)^N = \mathbf{1}$; hence T^-, the inverse of T^+, is equal to $(T^+)^{N-1}$. Now we introduce a spinor space $\mathcal{H}_s = \mathcal{C}^2$ to each points in $\mathcal{Z}_N$ and write trace of $End_{\mathcal{C}}(\mathcal{H}_s)$ as tr_s. A *fermion field* on $\mathcal{Z}_N$ is an element in $\mathcal{H} := \mathcal{A}(\mathcal{Z}_N) \otimes \mathcal{H}_s$. $\mathcal{H}$ appears to be a Hilbert space under a standard definition of inner product $(\psi, \psi') = \sum_{x=0}^{N-1} \sum_{i=1}^{2} \overline{\psi^i(x)} \psi'^i(x)$; and trace on $End_{\mathcal{C}}(\mathcal{H})$ is $Tr = Sp \cdot tr_s$. Under this inner product, the hermitian conjugate of T^+ is just T^-, i.e. $(T^+)^\dagger = T^-$.

Free Dirac operator acting on $\mathcal{H}$ is defined to be

$$F = \eta_+ + \eta_-$$

in which $\eta_\pm = T^\pm \sigma_\pm$ and $\sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. $\eta_\pm$ fulfil Clifford algebra relation on 2D-Euclidean space $\eta_\pm \eta_\pm = 0, \{\eta_\pm, \eta_\mp\} = \mathbf{1}$ and F is hertitian

$$F^\dagger = F \tag{2.1}$$

The fundamental *non-commutativity* in this formalism is $\eta_\pm f = (T^\pm f)\eta_\pm$. It is proved in [5] that Connes' distance on $\mathcal{Z}_N$ defined by

$$d_F(x, y) = sup\{|f(x) - f(y)| :\| [F, f] \| \leq 1\}, \forall x, y \in \mathcal{Z}_N$$

is just the conventional linear(Euclidean)-distance on $\mathcal{Z}_N$: $d(x, y) = min\{|x-y|, |N-x+y|\}$.

Gauge coupling can be introduced by a routine. First let $\mathcal{H}_c$ be a "color" space on which a gauge group G is represented as $R(G)$, and tensor-produce $\mathcal{H}$ by $\mathcal{H}_c$ forming a new Hilbert space $\widetilde{\mathcal{H}}$. A gauge transformation is a function $r : \mathcal{Z}_N \to R(G), x \mapsto r(x) \in R(G)$. Then a *covariant* Dirac operator is defined by

$$F(\omega) := \omega^\dagger \eta_+ + \eta_- \omega = \begin{pmatrix} & \omega^\dagger T^+ \\ T^- \omega & \end{pmatrix}$$

in which ω is the so-called *link variable*, a $End_{\mathcal{C}}(\mathcal{H}_c)$-valued function on $\mathcal{Z}_N$. The gauge transformation rule of $F(\omega)$ is

$$F(\omega')r = rF(\omega) \tag{2.2}$$

Hermiticity of $F(\omega)$ (Eq.(2.1)) is compatible with transformation rule (2.2), unless $[F(\omega), r^\dagger r] = 0$ [2]. In this paper we adopt the convention that R is unitary; accordingly, $r^\dagger r = \mathbf{1}$. The geometric significance of ω can be illustrated by

$$\omega'(x)r(x) = r(x+1)\omega(x), \omega'^\dagger(x)r(x+1) = r(x)\omega^\dagger(x),$$

namely, $\omega(x)$ is a *parallel transport* transporting $\mathcal{H}_c$ on x to $\mathcal{H}_c$ on x+1, while $\omega(x)^\dagger$ is another parallel transport transporting $\mathcal{H}_c$ on x+1 to $\mathcal{H}_c$ on x reversely. Here we just consider $\mathcal{H}_c = \mathcal{C}$, i.e. 1-dimensional case, in which $\omega \in \mathcal{A}(\mathcal{Z}_N)$. Hence, $\omega = \rho e^{i\theta}$ where ρ, θ are real functions, $\rho(x) \geq 0$ and $\omega^\dagger = \overline{\omega}$, $\omega^\dagger \omega = |\omega|^2 = \rho^2$. Adopt results in [5] [2], it is easy to verify that $d_{F(\omega)}(x, x+k)$ is equal to

$$min\{\frac{1}{\rho(x)}+\frac{1}{\rho(x+1)}+\ldots+\frac{1}{\rho(x+k-1)},\frac{1}{\rho(x-1)}+\frac{1}{\rho(x-2)}+\ldots+\frac{1}{\rho(x-(N-k))}\}$$

for all $x,k\in\mathcal{Z}_N$. Therefore, metric on $\mathcal{Z}_N$ is modified by the strength of parallel transports. This point can be illustrated more clearly by some special examples. 1)$\rho(x)=\rho_0>0$, then resulting metric differs from the "free" one by a lattice constant $1/\rho_0$; 2)$\rho(0)=0$, then $d_{F(\omega)}(0,1)=\infty$, which can be interpreted that the points $x=0$ and $x=1$ are disconnected; 3)$\rho(0)\to+\infty$, then $d_{F(\omega)}(0,1)\to 0$, which is interpreted as a "black hole" by M. Hale in [6].

In the next section, dynamics of ω will be considered; we will show how cases 1) and 2) will emerge from classical solutions to the equations of motion for ω. The remaining of this sector will deliver some necessary notations. First,

$$F(\omega)^2=T^-(\omega\omega^\dagger)\eta_-\eta_++(\omega^\dagger\omega)\eta_+\eta_-=T^-(\omega\omega^\dagger)\sigma_-\sigma_++(\omega^\dagger\omega)\sigma_+\sigma_- \tag{2.3}$$

Wedge product of $F(\omega)$ is defined by

$$F(\omega)\wedge F(\omega)=T^-(\omega\omega^\dagger)\eta_-\wedge\eta_++(\omega^\dagger\omega)\eta_+\wedge\eta_-=(T^-(\omega\omega^\dagger)-\omega^\dagger\omega)\sigma_-\wedge\sigma_+ \tag{2.4}$$

in which $\sigma_-\wedge\sigma_+=\frac{(-)}{2}\sigma_3$, $\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. So $F(\omega)\wedge F(\omega)=\frac{(-)}{2}\partial^-(\rho^2)\sigma_3$. Second, a "physical" Dirac operator is defined to be

$$\mathcal{D}(\omega):=\nabla^+\sigma_++\nabla^-\sigma_-$$

where $\nabla^-=T^-\omega-\mathbf{1}$, $\nabla^+=\omega^\dagger T^+-\mathbf{1}$. Consequently, $\mathcal{D}(\omega)=F(\omega)-\sigma_+-\sigma_-$,

$$\mathcal{D}(\omega)^2=\begin{pmatrix}\omega^\dagger\omega-T^-\omega-\omega^\dagger T^++\mathbf{1} & \\ & T^-(\omega\omega^\dagger)-T^-\omega-\omega^\dagger T^++\mathbf{1}\end{pmatrix} \tag{2.5}$$

and $\mathcal{D}(\omega)\wedge\mathcal{D}(\omega)=F(\omega)\wedge F(\omega)$.

III. ACTION FUNCTIONALS, EQUATIONS OF MOTION AND VACUUM SOLUTIONS

Several action functionals containing only ω will be considered below. What we mainly concern is whether there be non-trivial vacuum solutions in the corresponding equations of motion. By a "non-trivial vacuum", we mean that translation invariance of $\mathcal{Z}_N$ is broken by this vacuum.

A. Trivial Cases

1. $S_1[\omega]=Tr(F(\omega)^2)$

Notice Eq.(2.3), $S_1[\omega]=2\sum\omega^\dagger\omega=2\sum\rho^2$. This action admits only 0-solution obviously.

2. $S_2[\omega] = Tr(\mathcal{D}(\omega)^2)$

Notice Eq.(2.5), $S_2[\omega] = 2\sum(\omega^\dagger\omega + 1) = 2\sum(\rho^2 + 1)$, which admits only 0-solution either.

3. $S_3[\omega] = Tr(F(\omega)^4)$

Still by Eq.(2.3), $S_3[\omega] = 2\sum(\omega^\dagger\omega)^2 = 2\sum\rho^4$ admitting only 0-solution either.

4. $S_4[\omega] = Tr((F(\omega) \wedge F(\omega))(F(\omega) \wedge F(\omega)))$

By Eq.(2.4), $S_4[\omega] = \frac{1}{2}\sum(\partial^-(\rho^2))^2$. Equation of motion is deduced by $\delta_\omega S_4[\omega] = 0$

$$\rho(x)(2\rho(x)^2 - \rho(x+1)^2 - \rho(x-1)^2) = 0, \forall x \in \mathcal{Z}_N \tag{3.1}$$

If $\rho(x) \neq 0, \forall x \in \mathcal{Z}$, let $\phi = \rho^2$, Eq.(3.1) takes on the form

$$\partial^+\partial^-\phi = 0 \tag{3.2}$$

which is a discretized harmonic equation $\Delta\phi = 0$ on S^1. Eq.(3.2) admits only constant solution $\rho(x) = \rho_0$, which can be understood schematically by the *extreme value principle* in commutative harmonic analysis [7]. The "on-shell" metric satisfies $d_{F(\omega)}(,) = \frac{1}{\rho_0}d_F(,)$ which corresponds the case 1) in the last section, namely that lattice constant gains a "renormalization" from 1 to $1/\rho_0$.

Else if there is a x_0 such that $\rho(x_0) = 0$, then the only consistent solution is $\rho(x) = 0$ for all x.

5. $S_5[\omega] = Tr(\mathcal{D}(\omega)^4)$

From Eq.(2.5), $S_5[\omega] = 2\sum(\rho^4 + 2\rho^2 + 1)$. The equation of motion $\rho(\rho^2 + 1) = 0$ admits only 0-solution.

B. Nontrivial Case

Now consider

$$S[\omega] = \frac{1}{2}Tr((F(\omega) \wedge F(\omega))(F(\omega) \wedge F(\omega)) + \frac{\alpha}{4}F(\omega)^4 - \frac{\beta}{2}F(\omega)^2) \tag{3.3}$$

in which coupling constants $\alpha, \beta \in \mathcal{R}$ and will not equal to zero at the same time. One can check that

$$S[\omega] = \sum(\frac{1}{4}(\partial^-(\rho^2))^2 + \frac{\alpha}{4}\rho^4 - \frac{\beta}{2}\rho^2)$$

The equation of motion is given by

$$\rho(\partial^+\partial^-(\rho^2)+\alpha\rho^2-\beta)=0$$

or equivalently

$$\rho(t\rho^2-T^+(\rho^2)-T^-(\rho^2)-\beta)=0 \tag{3.4}$$

where $t:=2+\alpha$.

1. Non-singular Case: $\rho(x)>0, \forall x\in\mathcal{Z}_N$

Remember the definition of ϕ, then Eq.(3.4) changes form as

$$-\phi(x-1)+t\phi(x)-\phi(x+1)=\beta, \forall x\in\mathcal{Z}_N$$

or in a matrix form equivalently

$$\begin{pmatrix} t & -1 & 0 & 0 & \dots & 0 & 0 & -1 \\ -1 & t & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & t & -1 & \dots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & t & -1 \\ -1 & 0 & 0 & 0 & \dots & 0 & -1 & t \end{pmatrix} \begin{pmatrix} \phi(0) \\ \phi(1) \\ \phi(2) \\ \vdots \\ \phi(N-2) \\ \phi(N-1) \end{pmatrix} = \beta \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{pmatrix} \tag{3.5}$$

The cyclic symmetry of Eq.(3.5) $\phi(x)\to\phi(x+1)$ implies that $\phi(x)=\phi_0, \forall x\in\mathcal{Z}_N$ in which $\phi_0=\frac{\beta}{t-2}=\frac{\beta}{\alpha}$. Note that if $\alpha=0, \beta\neq 0$, no solution exists. Upon this solution as a background, metric fulfills relation $d_{F(\omega)}(,)=\sqrt{\frac{\alpha}{\beta}}d_F(,)$.

2. Singular Case

The remainder of this section will be devoted to the most interesting situation. Let's first consider a general case. Denote $R_0:=\{x\in\mathcal{Z}_N:\rho(x)=0\}$ and suppose that R_0 is not empty, then $\mathcal{Z}_N$ is divided by R_0 into a collection of subsets $S_a\subset\mathcal{Z}_N$ where each $S_a=\{x_a,x_a+1,x_a+2,\dots,x_a+l_a\}$ with $l_a\geq 1$, $\rho(x_a-1)=0$, $\rho(x_a+l_a)=0$, $\rho(x_a+k)\neq 0, k=0,1,\dots,l_a-1$. It is easy to imagine that $d_{F(\omega)}(x,y)\to\infty$, if $x\in S_a, y\in S_b, a\neq b$, i.e. case 2) in the last section. Therefore, R_0 cuts $\mathcal{Z}_N$ into disconnected segments S_a. Now we consider R_0 contains only one point and without losing generality, we set $\rho(0)=0, \rho(x)>0, x=1,2,\dots N-1$. If we find out a solution to this case, then this solution breaks translation invariance, hence being a nontrivial vacuum.

We have to prepare some matrix algebra here. Introduce a matrix sequence $\{M_n(t):n=1,2,...\}$ where

$$M_n(t):=\begin{pmatrix} t & -1 & 0 & 0 & \dots & 0 & 0 & 0 \\ -1 & t & -1 & 0 & \dots & 0 & 0 & 0 \\ 0 & -1 & t & -1 & \dots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & \dots & -1 & t & -1 \\ 0 & 0 & 0 & 0 & \dots & 0 & -1 & t \end{pmatrix}$$

is a n×n matrix, and define

$$F_n(t) = det(M_n(t)) \tag{3.6}$$

One can prove this iterative relation easily

$$F_0(t) := 1, F_1(t) = t, F_n(t) = tF_{n-1}(t) - F_{n-2}(t), n = 2, 3, ... \tag{3.7}$$

If use a convention in mathematical physics $t = 2x$, then Eq.(3.7) is the definition of *Chebyshev's polynomials of the second kind* [8]. So definition (3.6) provides Chebyshev's polynomials with another interpretation. Expressions of the first a few $F_n(t), n = 0, 1, 2, \ldots, 10$ are put in Appendix A; and plots of $F_n(t), n = 2, 3, ..., 10$ taken as function of $t \in [0, 3]$ is showed in Figure 1. It is clearly indicated from this figure that all real roots of these $F_n(t)$ distribute in $[-2, 2]$. We will show this statement is rigid for any n below. It can be deduced from Eq.(3.7) directly $F_n(2) = n+1$. The solution of Eq.(3.7) can be written in two different form:

$$F_{2m}(t) = \sum_{k=0}^{m} (-)^{m-k} \binom{m+k}{m-k} t^{2k}, F_{2m+1}(t) = \sum_{k=0}^{m} (-)^{m-k} \binom{m+1+k}{m-k} t^{2k+1}$$

or

$$F_n(t) = \frac{z_+^{n+1} - z_-^{n+1}}{\sqrt{t^2 - 4}} \tag{3.8}$$

in which $z_\pm = (t \pm \sqrt{t^2 - 4})/2$. From Eq.(3.8), it is easy to verify that $F_n(t) > 0, n = 0, 1, 2, \ldots,$ if $t > 2$. This is a critically important property that we will make use of. Set $t > 2$, i.e. $\alpha > 0$ below.

Now let $\Phi = (\phi(1), \phi(2), \ldots, \phi(N-1))^T$, $\mathbf{1}_V = (1, 1, \ldots, 1)^T$, then equation of motion (3.4) can be written as

$$M_{N-1}(t)\Phi = \beta \mathbf{1}_V \tag{3.9}$$

Physical solution to Eq.(3.9) must be positive-definite. Take $\beta \neq 0$ to be a dimensional parameter and define dimension-less variables $\phi = \beta v, \Phi = \beta V$. Then

$$M_{N-1}(t)V = \mathbf{1}_V \tag{3.10}$$

Introduce notations $f_{i,j}(t) = \frac{F_i(t)}{F_j(t)}$, $\Sigma_n(t) = \sum_{j=0}^{n} F_j(t)$, $\psi_n = \frac{\Sigma_n(t)}{F_{n+1}(t)}$. Eq.(3.10) can be solved by iterative relations $v(1) = f_{1,2}(t)v(2) + \psi_1(t)$, $v(2) = f_{2,3}(t)v(3) + \psi_2(t)$, ..., $v(x) = f_{x,x+1}(t)v(x+1) + \psi_x(t)$, ..., $v(N-1) = \psi_{N-1}(t)$. The formal linear solution is

$$v(x) = \sum_{y=0}^{N-1-x} f_{x,x+y}(t)\psi_{x+y}(t), x = 1, 2, ..., N-1 \tag{3.11}$$

Since we choose $t > 2$, there is no singularity in $v(x)$. From the inversion symmetry $v(x) \leftrightarrow v(N-1-x)$, one can get the identity

$$\psi_x(t) + f_{x,x+1}(t)\psi_{x+1}(t) + \ldots + f_{k,N-1}(t)\psi_{N-1}(t) =$$

$$\psi_{N-1-x}(t) + f_{N-1-x,N-x}(t)\psi_{N-x}(t) + \ldots + f_{N-1-x,N-1}(t)\psi_{N-1}(t)$$

Now we consider the positivity of ϕ. In fact, if $\beta > 0$, due to the positivity of $F_n(t)$ when $t > 2(\alpha > 0)$, there is that $\phi(x) > 0, \forall x = 1, 2, ..., N-1$. Solution in cases $N = 3, 4, 5, 6$ are demonstrated in Appendix B. The physical interpretation of this solution is that when coupling $\alpha > 0$ and $\beta > 0$, there is a nontrivial vacuum appearing and that the metric appears to be

$$d_{F(\omega)}(x, y) = \frac{1}{\sqrt{\beta}}\left(\frac{1}{\sqrt{v(x)}} + \frac{1}{\sqrt{v(x_1)}} + \ldots + \frac{1}{\sqrt{v(y-1)}}\right)$$

for all $x, y = 1, 2, ..., N-1, x \leq y$. Moreover, this statement is valid in the limit $N \to \infty$.

IV. DISCUSSIONS

Till now we do not consider the problem of dimensions. In fact, by introducing a lattice constant a and a "Newtonian Constant" G, action (3.3) can be put into a dimensional form

$$S[\omega] = \frac{a}{2G} Tr((F(\omega) \wedge F(\omega))(F(\omega) \wedge F(\omega)) + \frac{\alpha}{4} F(\omega)^4 - \frac{\beta}{2} F(\omega)^2)$$

in which the dimensions are assigned as $[\omega] = L^{-1}$, $[G] = L^{-3}$, $[a] = L$, $[\alpha] = L^0$, $[\beta] = L^{-2}$. Consequently, $d_{F(\omega)}(,)$ has the correct length dimension and action $S[\omega]$ is dimension-less.

We have not discussed the interaction between link variable(connections, parallel transports) and matter fields. Work on this aspect is in proceeding.

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APPENDIX A

Expressions of $F_n(t), n = 1, 2, ..., 10$

$$F_0(t) = 1$$

$$F_1(t) = t$$

$$F_2(t) = t^2 - 1$$

$$F_3(t) = t^3 - 2t$$

$$F_4(t) = t^4 - 3t^2 + 1$$

$$F_5(t) = t^5 - 4t^3 + 3t$$

$$F_6(t) = t^6 - 5t^4 + 6t^2 - 1$$

$$F_7(t) = t^7 - 6t^5 + 10t^3 - 4t$$

$$F_8(t) = t^8 - 7t^6 + 15t^4 - 10t^2 + 1$$

$$F_9(t) = t^9 - 8t^7 + 21t^5 - 20t^3 + 5t$$

$$F_{10}(t) = t^{10} - 9t^8 + 28t^6 - 35t^4 + 15t^2 - 1$$

APPENDIX B

Solutions to Eq.(3.10)

$$N = 3 : v(1) = v(2) = \frac{1}{t-1}$$

$$N = 4 : v(1) = v(3) = \frac{t+1}{t^2-2}; v(2) = \frac{t+2}{t^2-2}$$

$$N = 5 : v(1) = v(4) = \frac{t}{t^2-t-1}; v(2) = v(3) = \frac{t+1}{t^2-t-1}$$

$$N = 6 : v(1) = v(5) = \frac{t^2+t-1}{t(t^2-3)}; v(2) = v(4) = \frac{t+2}{t^2-3}; v(3) = \frac{t^2+2t+1}{t(t^2-3)}$$

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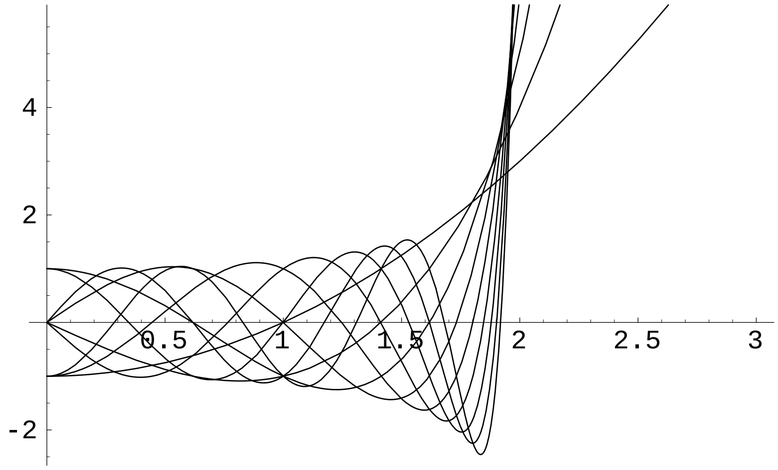

FIG. 1. $F_n(t), n = 2, 3, ..., 10,\ t \in [0, 3]$